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NAVAL POSTGRADUATE SCHOOL

Monterey, California

CONTROLLING THE INFLUENCES

OF COMPONENT VARIABLES

by

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ABSTRACT

In the comparison of military units or systems, many attributes, including performance, might be measured. One general approach to determining which system is best involves forming a composite measure of the differentially weighted component measures. It is a fairly common practice to control the component means and variances; it is less common, and more difficult, to control component covariances and, thus, to control the contribution of a component to the variance of the composite measure (variable). This paper presents a method leading to a computer procedure for assigning equal or differential influences on the composite by the use of component means, standard deviations, and covariances. Two numerical examples are given herein, one illustrating equal and the other unequal weighting. The technique appears to be both fast and precise.

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I. INTRODUCTION

The section following this shows that the amount of variance a component contributes to a set of composite scores is a function of the component's variance and its covariances with the other components. If a component's original covariances are low, it may be an indication that this variable is measuring attributes different from those being measured by the remaining component variables, or that it is measuring the same attributes less reliably. By using the procedure and assumption described in this paper, a researcher can equalize the contributions of all of the components to the variance of the set of composite scores. Before doing so, however, he should have evidence available which leads him to conclude that it is necessary to increase the contribution of a variable having low covariance terms. If, for example, the component having low covariances consists of a set of judgments made by any individual rater, the researcher should be convinced that this rater was judging relevant attributes of behavior ignored by other raters, or that he is not, instead, contributing non-valid variance to the composite scores. If the latter is true, increasing the effect of these data by using the method presented in this paper is, at best, unfortunate.

1. This report is an extension of an earlier article by R. S. Elster and C. O. Nystrom, titled, "A computerized Method for Controlling Components' Contributions to the Variance of a Composite", appearing in *Educational and Psychological Measurement*, Vol. 30, No. 3, Autumn 1970.

II. DERIVATION AND COMPUTATIONAL SOLUTION

Beginning with an initial data matrix, the necessary equations can be developed. In general terms, we usually have an $n \times k$ score matrix B, giving the scores of n units (or some other objects of evaluation) on k variables (components). This matrix is then standardized so that each component's observations have a mean of zero and a variance of one; this is designated as matrix Z. Matrix Z is also $n \times k$.

Before applying weights to any of the k component scores in Z, a composite score, C_i, may be computed for each of the n evaluated objects by using:

$$C_i = \sum_{j=1}^{k} Z_{ij}, \text{ e.g., } C_1 = \sum_{j=1}^{k} Z_{1j} \tag{1}$$

The variance of the array of C_i's is given by:

$$\begin{aligned} S_c^2 = S_1^2 &+ S_1 S_2 r_{12} + \cdots + S_1 S_k r_{1k} + \\ S_2^2 &+ S_2 S_1 r_{21} + \cdots + S_2 S_k r_{2k} + \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots + \\ S_k^2 &+ S_k S_1 r_{k1} + \cdots + S_k S_{k-1} r_{k(k-1)} , \end{aligned} \tag{2}$$

where $r_{k(k-1)}$ designates the correlation between components k and

(k-1). For standard scores having a mean of zero and a variance of one, the variance of the array of C_i's is given by:

$$
\begin{aligned}
S_c^2 = 1 + r_{12} + \cdots\cdots + r_{1k} &+ \\
1 + r_{21} + \cdots\cdots + r_{2k} &+ \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots &+ \\
1 + r_{k1} + \cdots\cdots + r_{k(k-1)} &
\end{aligned} \tag{3}
$$

In equation (3) is a series of terms for each component consisting of the component's variance along with its correlations with all of the other components. As can be seen from equation (3), the contribution of the jth component to the composite score variance is given by:

$$\mathrm{Var}\,(C_j) = 1 + \sum_{\substack{m=1 \\ m \neq j}}^{k} r_{jm} \tag{4}$$

In order to accomplish the goal of controlling the proportion of the composite's variance contributed by each of the components, one obviously must be able to adjust the magnitudes of their variances. The adjustment method to be used here is similar to that used by Dunnette and Hoggatt (1957, pp. 430-434): we shall multiply the set of component scores by a set of weights; the weights will have values so that the magnitudes of their new variances will equal the proportions of variance we wish the separate components to contribute to the composite.

Designating the weight for the jth component by A_j, the new composite may be computed by:

$$C_i' = \sum_{j=1}^{k} A_j Z_{ij}, \text{ e.g., } C_1' = \sum_{j=1}^{k} A_j Z_{1j} \tag{5}$$

If the components are in standard score form, the variance of the C_i' 's is given by:

$$\text{Var}\,(C_i') = \sum_{j=1}^{k} A_j^2 + \sum_{j=1}^{k} \sum_{\substack{m=1 \\ m \neq j}}^{k} A_j A_m r_{j\,m} \tag{6}$$

and the contribution, b_j, of the jth component to the composite score variance is given by:

$$b_j = A_j^2 + \sum_{\substack{m=1 \\ m \neq j}}^{k} A_j A_m r_{jm} \tag{7}$$

An equation such as (7) can be written for each of the j components. Given this set of j equations, and given that b_j designates the variance we wish component j to contribute to the composite, the following system of quadratic equations must be solved.

$$A_j^2 + \sum_{\substack{m=1 \\ m \neq j}}^{k} A_j A_m r_{jm} = b_j \tag{8}$$

$$j = 1, 2, \ldots, k$$

The solution to the above set of equations will yield the set of weights (the A_j's) to be used as multipliers with the k components so that the desired values of the b_j's are obtained.

The procedure used for solving the system of simultaneous quadratic equations shown in equation (8) is the Newton-Raphson method as shown by Scarborough (1962, pp. 213-217).

The computational procedure has been programmed as a subroutine. To use it, the practitioner need only supply: 1) the original score matrix, giving the scores of n persons on the j components; 2) the proportion, b_j, that each of the j components is to contribute to the variance of the composite (each b_j must be expressed as a positive decimal, and the sum of the b's must be equal to one); and 3) an ε value that terminates the iterative computational procedure. The parameter ε will be discussed later in this report.

III. MATHEMATICS OF THE COMPUTATIONAL PROCEDURE

In the case of two components, there would be two equations in two unknowns. Let $\phi(X,Y) = 0$, and $\psi(X,Y) = 0$ represent the two equations. If X_0, Y_0 are approximate roots of ϕ and ψ, and h and ℓ are corrections, then let

$$X = X_0 + h$$

$$Y = Y_0 + \ell$$

Scarborough (1962, pp. 214) shows that

$$\phi(X_0,Y_0) + h(\delta\phi/\delta x)_0 + \ell(\delta\phi/\delta y)_0 = 0$$

$$\psi(X_0,Y_0) + h(\delta\psi/\delta x)_0 + \ell(\delta\psi/\delta y)_0 = 0$$

or

$$\begin{vmatrix} (\delta\phi/\delta x)_0 & (\delta\phi/\delta y)_0 \\ (\delta\psi/\delta x)_0 & (\delta\psi/\delta y)_0 \end{vmatrix} \begin{vmatrix} h \\ \ell \end{vmatrix} \begin{matrix} = \\ = \end{matrix} \begin{vmatrix} -\phi(X_0,Y_0) \\ -\psi(X_0,Y_0) \end{vmatrix} \tag{9}$$

The goal is to solve for h and ℓ letting $X_1 = X_0 + h$, $Y_1 = Y_0 + \ell$, etc.; iterating until a desired level of accuracy is achieved.

In the general case of weighting components which are in standard score form (using the notation from equation (8)):

$$\phi_j = A_j \sum_{m=1}^{k} A_m r_{jm} - b_j \; ; \; j = 1, 2, \ldots, K \tag{10}$$

Then, letting $\phi_j^* = \phi_j/A_j$, and differentiating, $\delta\phi_j^*/\delta A_m = r_{jm}$, when $j \neq m$ and $\delta\phi_j^*/\delta A_m = b_j/A_j^2 + 1$, when $j = m$.

So for the general case, Equation (9) becomes

$$\begin{vmatrix} 1+\dfrac{b_1^{(t)}}{\left(A_1^{(t)}\right)^2} & r_{12} & r_{13} & \cdots & r_{1k} \\ r_{21} & 1+\dfrac{b_2^{(t)}}{\left(A_2^{(t)}\right)^2} & r_{23} & \cdots & r_{2k} \\ \cdot & \cdot & \cdot & \cdots & \\ \cdot & \cdot & \cdot & \cdots & \\ \cdot & \cdot & \cdot & \cdots & \\ r_{k1} & r_{k2} & \cdots & \cdot & 1+\dfrac{b_k^{(t)}}{\left(A_k^{(t)}\right)^2} \end{vmatrix} \begin{vmatrix} h_1 \\ h_2 \\ \cdot \\ \cdot \\ \cdot \\ h_k \end{vmatrix} = \begin{vmatrix} \dfrac{b_1^{(t)}}{A_1^{(t)}} - \sum_{j=1}^{k} A_j^{(t)} r_{j1} \\ \dfrac{b_2^{(t)}}{A_2^{(t)}} - \sum_{j=1}^{k} A_j^{(t)} r_{j2} \\ \cdot \\ \cdot \\ \cdot \\ \dfrac{b_k^{(t)}}{A_k^{(t)}} - \sum_{j=1}^{k} A_k^{(t)} r_{jk} \end{vmatrix} \qquad (11)$$

The iterations are performed by letting $A_j^{(t+1)} = A_j^{(t)} + h_j$, $j = 1, 2, \ldots, k$, and recalculating both the right-hand array and the diagonal of the coefficient array. For an exact solution, the right-hand side of Equation (11) will go term by term to zero. The convergence procedure used by the author of this paper tested to determine if the terms on the right side of Equation (11) had all become equal to, or less than, an epsilon (ε). The user must specify an ε value which is used in terminating the iterations when all of the terms on the right-hand side of Equation (11) are $<\varepsilon$. Initially,

$$A_j^{(o)} = 1.0 \text{ for all } j .$$

IV. EXAMPLES

Artificial raw scores were generated yielding the standard score covariance matrix given in Table 1. (This matrix is an optional output available from the computer program allowing the user to examine the original component weights.)

By examining the row of figures in Table 1 giving the percent contribution of each rater (component) to the variance of the composite scores, it is evident that the original components are unequally weighted. Using the data given in the table, two solutions are demonstrated in the following paragraphs. The first example illustrates the case in which the user wishes to assign equal weights to all of the components. In the second example, the user assigns unequal proportions to the components.

Table 1 STANDARD SCORE COVARIANCE MATRIX

Raters (Components)	Raters (Components) A	B	C	D	E	F
A	1.00	-.225	.233	-.084	.118	-.509
B	-.225	1.00	-.009	.521	.099	-.151
C	.233	-.009	1.00	.112	.256	-.306
D	-.084	.521	.112	1.00	.196	-.264
E	.118	.099	.256	.196	1.00	-.236
F	-.509	-.151	-.306	-.264	-.236	1.00
Rater's Contribution to the composite score variance	.533	1.235	1.286	1.481	1.433	-.466

Total score variance = 5.502

A. Equal Weighting

The data summarized in Table 1 were initially run through the computational subroutine with the specification that each of the six components should make an identical contribution to the variance of the composite. The weights to be applied to the components' standard scores in order to equate the contributions of the six components are given in Table 2. The weights in Table 2 were derived by applying the computational method outlined above to the system of six equations similar to that given in Equation (8); the weights in Table 2 are the roots of the six equations for this example.

Table 2 WEIGHTS TO BE USED AS MULTIPLIERS WITH THE COMPONENTS' STANDARD SCORES FOR THE EQUAL CONTRIBUTION EXAMPLE

Raters (Components)	A	B	C	D	E	F
Weights	1.638	1.031	.937	.904	.858	2.144

Table 3 shows the covariance matrix that resulted after applying these weights to the components' standard scores. Table 3 (using the notation of Equation 8) contains the elements $A_j2\ r_{jj}$ on the main diagonal and the elements $A_j\ A_m\ r_{jm}$ off the diagonal.

The column in Table 3 that lists the percentage of the composite's variance contributed by each of the components shows that the contributions of the six components were equated by the computational method. The solution of the equations required 4 iterations; about 3.2 seconds of computer time was required for the entire run.

(The iterations were terminated when each of the six roots had an accuracy of $\leq .00001$.)

B. Unequal Weighting

To illustrate that the computatonal procedure can also be used in solving for differential weights, Tables 4 and 5 show the results obtained when the six components were specified as contributing .50, .10, .10, .10, and .10 of the composite's variance. This run required 4 iterations to achieve $\leq .00001$ accuracy for each of the six roots.

Table 3 COVARIANCE MATRIX DERIVED AFTER APPLYING THE COMPUTED WEIGHTS FOR THE EQUAL CONTRIBUTION EXAMPLE

Raters (Components)	Raters (Components) A	B	C	D	E	F	Per Cent[a]
A	2.683	-.380	.357	-.125	.166	-1.788	16.67
B	-.380	1.063	-.008	.485	.087	-.334	16.67
C	.357	-.008	.878	.095	.206	-.615	16.67
D	-.125	.485	.095	.817	.152	-.512	16.65
E	.166	.087	.206	.152	.736	-.433	16.68
F	-1.788	-.334	-.615	-.512	-.433	4.595	16.67
Rater's Contribution to the composite score variance	.913	.913	.913	.912	.914	.913	100.01[b]

Total score variance = 5.478

[a]Percent the component contributed to the composite's variance

[b]This column does not total to 100.00 due to rounding errors.

Table 4 WEIGHTS TO BE USED AS MULTIPLIERS WITH THE COMPONENTS' STANDARD SCORES FOR THE UNEQUAL CONTRIBUTION EXAMPLE

Raters (Components	A	B	C	D	E	F
Weights	2.324	.960	.680	.772	.654	2.151

Table 5 COVARIANCE MATRIX DERIVED AFTER APPLYING THE COMPUTED WEIGHTS FOR THE UNEQUAL CONTRIBUTION EXAMPLE

Raters (Components)	Raters (Components) A	B	C	D	E	F	Per Cent[a]
A	5.402	-.503	.368	-.151	.180	-2.546	49.99
B	-.503	.922	-.005	.386	.062	-.312	9.98
C	.368	-.006	.463	.059	.114	-.448	10.00
D	-.151	.386	.059	.596	.099	-.438	10.02
E	.180	.062	.113	.099	.427	-.331	10.02
F	-2.546	-.312	-.448	-.438	-.331	4.625	10.00
Rater's Contribution to the composite score variance	2.750	.549	.550	.551	.551	.550	100.01[b]

Total score variance = 5.501

[a]Per cent the component contributed to the composite's variance

[b]This column does not total to 100.00 due to rounding errors.

V. SUMMARY

A computer procedure was developed for weighting components in a composite. The procedure allows the user to freely choose how much of the variance of a set of composite scores is contributed by each of a number of components.

Two numerical examples using artificial input data were demonstrated illustrating an equal weighting solution and an unequal weighting solution. In both sample problems, the solutions reached the desired level of accuracy in about three seconds.

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13. ABSTRACT

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